Journal of Engineering Physics and Thermophysics, Vol. 75, No. 6, 2002

MOTION OF A VISCOUS INCOMPRESSIBLE FLUID IN A PIPE IN A ROTATING COORDINATE SYSTEM

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UDC 532

An analytical solution of the problem on steady motion of a viscous incompressible fluid in a circular pipe has been obtained with regard for the action of Coriolis forces. It is shown that the rotation of the coordinate system as a result of the destruction of the laminar regime of flow and the appearance of a partially nonuniform three-dimensional vortex flow on its basis lead to an increase in the hydrodynamic drag of the pipe.

Many important technical applications of hydrodynamics concern determination of the parameters of a fluid moving in noninertial rotating coordinate systems. Flow of gases in the gap between the blades of a turbine rotor or a pump and motion of a fluid in pipelines of a moving transport, an airplane, or a ship can serve as clear examples of this kind. Such problems have found dramatic embodiment in investigations of atmospheric and oceanic flows where the rotation of the earth determines many usual natural phenomena and well-known features of oceanic circulation [1, 2]. Although the theory of rotating liquids has been developed fairly well by now [3–5], in the scientific literature there are actually no data on the characteristics of flow of a viscous fluid in pipes and channels allowing for the additional action of inertial forces on the flow.

Interest in this problem has been provoked by the fact that the action of inertial forces in fluid flow in pipes can manifest itself even at fairly low angular velocities of rotation of the coordinate system. The reason is that in fluid flow in a pipe not only the transverse (transversal) component of the velocity is disturbed, but its longitudinal component also changes as a result of the displacement of particles under the action of Coriolis forces. In this case, as the calculations show, the longitudinal velocity component can be disturbed at much smaller distances from the initial portion of the pipe than the transverse component. Indeed, for a fluid particle moving with a velocity U along the pipe the change in the transverse velocity will be equal in order of magnitude to $\Delta v = 2\omega U \Delta t$ and the displacement in the transverse direction will be $\Delta r = \omega U(\Delta t)^2$ as a result of the Coriolis acceleration. Taking into account the fact that $\Delta t = L/U$, for the laminar flow in a circular pipe with the parabolic velocity distribution $U = 2V_0(1 - r^2)$ it is easy to obtain the following estimates of the change in the transverse v and longitudinal w components of the velocity as a function of the length L of the portion of the pipe:

$$\frac{\Delta v}{V_0} = \frac{2\omega L}{V_0}, \quad \frac{\Delta w}{V_0} = \frac{8\omega R_0}{V_0} \frac{\frac{r}{R_0}}{1 - \left(\frac{r}{R_0}\right)^2} \left(\frac{L}{2R_0}\right)^2 .$$

By way of example, we evaluate the influence of the rotation of the earth on the change in the velocity of flow in a circular pipe ($\omega = 7.27 \cdot 10^{-5}$ rad/sec). Let us assume that $V_0 = 1$ m/sec and $R_0 = 0.025$ m. Then in the first case, according to the above relations, the initial portion must be equal to about 10^5 diameters when the transverse velocity component changes by 5%, whereas it is equal only to 30–50 calibers for the same change in the longitudinal component of the velocity in the near-wall region of flow. The estimate of the initial-portion length equal to 30–50 calibers, obtained in the second case, gives grounds to expect that even at such low angular velocities of rotation of the coordinate system the structure of fluid flow can undergo certain changes.

The solution of the problem on steady motion of an incompressible fluid in a circular pipe under the additional action of Coriolis forces on the flow is more difficult mathematically than the solution of the analogous problem

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for a plane channel because of the nonlinearity and two-dimensionality of the initial equations [6]. In this case, more complex is the dynamics of generation of secondary flows when laminar translatory motion gives way to a more complex three-dimensional pattern of interaction of different-scale vortices with shear flows changing in value and direction. To describe this phenomenon completely it is necessary to use numerical methods. However, the problem is difficult even for numerical calculations with the use of a computer because of the complex spatial structure of the flow. Therefore, along with numerical calculations, of certain interest are approximate analytical solutions that allow one to determine the most important characteristics of flow by comparatively simple methods.

The system of initial equations in dimensionless form is written as follows:

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \varphi}-\frac{v^2}{r}=-\,\mathrm{Eu}\,\frac{\partial p}{\partial r}+\frac{1}{\mathrm{Re}}\left(\nabla^2 u-\frac{2}{r^2}\frac{\partial v}{\partial \varphi}-\frac{u}{r^2}\right)+\frac{2\omega R_0}{V_0}\,w\sin\varphi\,;$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \varphi}+\frac{uv}{r}=-\,\mathrm{Eu}\,\frac{1}{r}\frac{\partial p}{\partial \varphi}+\frac{1}{\mathrm{Re}}\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial \varphi}-\frac{v}{r^2}\right)+\frac{2\omega R_0}{V_0}\,w\cos\varphi\,; \qquad (1)$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \varphi}=-\,\mathrm{Eu}\,\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\nabla^2 w+\frac{2\omega R_0}{V_0}\left(u\sin\varphi+v\cos\varphi\right);\quad \frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial \varphi}=0\,,$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\,.$$

We will consider fluid flow at small angular velocities of rotation of the coordinate system. The problem will be solved by the method of expansion in a small parameter, for which the dimensionless relation $\varepsilon = \omega R_0/V_0$ reverse to the Rossby number will be used. Let us represent the functions sought in the form of series:

$$u=\sum_{n=1}^{\infty}u_n(r,\varphi)\,\varepsilon^n\,;\quad v=\sum_{n=1}^{\infty}v_n(r,\varphi)\,\varepsilon^n\,;\quad w=w_0+\sum_{n=1}^{\infty}w_n(r,\varphi)\,\varepsilon^n\,;\quad p=p_0+\sum_{n=1}^{\infty}v_n(r,\varphi)\,\varepsilon^n\,, \qquad (2)$$

where, as the zero approximation, we take the known parabolic velocity distribution in a circular pipe [7]

$$w_0=2\,(1-r^2)\,.$$

Substituting relations (2) into (1) and equating the coefficients ε of the same powers to zero, we obtain a chain of equations for successive determination of the functions sought in expansion (2).

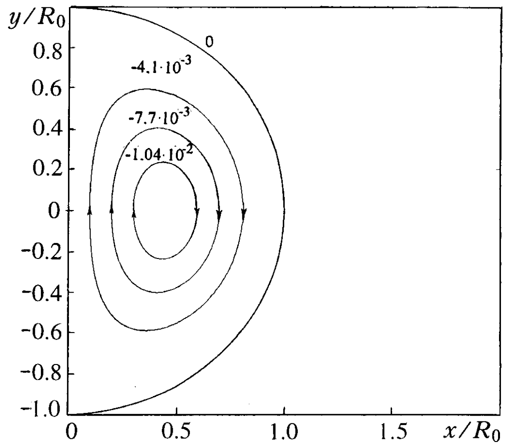
The system of differential equations of the first approximation is written in the form

$$\frac{1}{\mathrm{Re}}\left(\nabla^2 u_1-\frac{2}{r^2}\frac{\partial v_1}{\partial \varphi}-\frac{u_1}{r^2}\right)+2w_0\sin\varphi-\mathrm{Eu}\frac{\partial p_1}{\partial r}=0\,;$$

$$\frac{1}{\mathrm{Re}}\left(\nabla^2 v_1+\frac{2}{r^2}\frac{\partial u_1}{\partial \varphi}-\frac{v_1}{r^2}\right)+2w_0\cos\varphi-\mathrm{Eu}\frac{1}{r}\frac{\partial p_1}{\partial \varphi}=0\,;\quad \frac{\partial}{\partial r}(ru_1)+\frac{\partial v_1}{\partial \varphi}=0\,.$$

With zero boundary conditions on the pipe wall it has the solution

$$\varepsilon u_1=\frac{1}{24}\frac{\omega R_0^2}{\nu}(r^4-2r^2+1)\sin\varphi\,;\quad \varepsilon v_1=\frac{1}{24}\frac{\omega R_0^2}{\nu}(5r^4-6r^2+1)\cos\varphi\,;$$



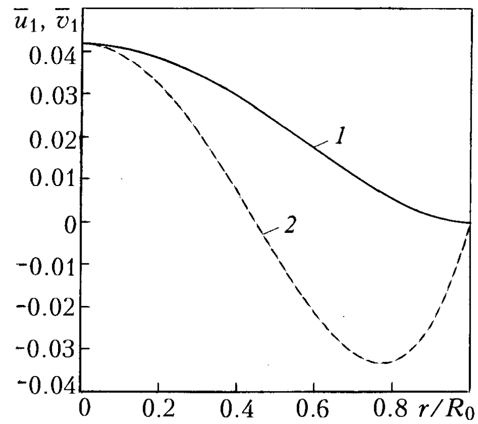


Fig. 1. Pattern of flow in the cross section of a pipe for different values of the stream function $\overline{\psi}_1 = \psi_1 \nu / (\omega R_0^2)$.

Fig. 2. Profiles of radial (1) and transversal (2) velocities of transverse flow in the first approximation.

$$\varepsilon p_1 = \frac{1}{24} \frac{\omega R_0^2}{\nu} (-3r^3 + 10) \sin \varphi \, ; \quad \varepsilon \psi_1 = -\frac{1}{24} \frac{\omega R_0^2}{\nu} r (r^4 - 2r^2 + 1) \cos \varphi \, . \tag{3}$$

To a first approximation, disturbed fluid motion represents a system of two vortices rotating in opposite directions and positioned symmetrically relative to the OY axis. The vortex in the right semicircle rotates clockwise, while the vortex in the left semicircle rotates in the opposite direction. The intensity of rotation is proportional to the value of the dimensionless combination $\omega R_0^2 / \nu = 1/E$. Figure 1 shows the streamlines of one such vortex positioned in the right half-plane.

The profiles of the radial $\overline{u}_1 = \varepsilon u_1 \nu / (\omega R_0^2 \sin \varphi)$ and transversal $\overline{v}_1 = \varepsilon v_1 \nu / (\omega R_0^2 \cos \varphi)$ velocities of this motion are presented in Fig. 2. As is clear from the figure, the maximum value of the transverse velocity component is attained at the center of the pipe. Its gradient near the wall differs from zero. As a result of the rotation of the vortices, there arise additional shear stresses that lead to a loss in the mechanical energy and, as a consequence, to an increase in the hydrodynamic drag of the pipe. For this reason, the hydrodynamic drag in the rotating coordinate system is always higher than the hydrodynamic drag of the pipe under usual conditions. It is of interest to note that the vortex rotation of the fluid leads to the appearance of an additional friction force acting on the pipe in the transverse direction (in the negative direction of the OY axis in the considered case). This force is applied oppositely to the Coriolis force and is caused by the reverse transverse fluid flow near the pipe wall.

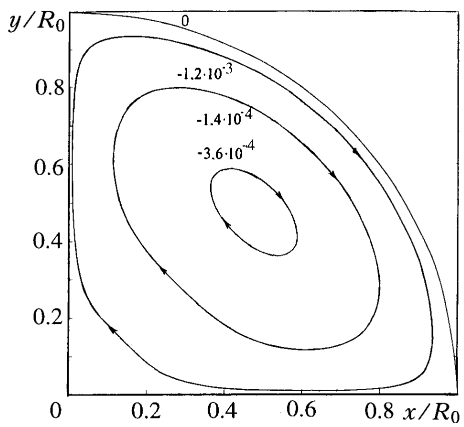
The system of differential equations of the second approximation is written as follows:

$$\frac{1}{\mathrm{Re}} \left(\nabla^2 u_2 - \frac{2}{r^2} \frac{\partial v_2}{\partial \varphi} - \frac{u_2}{r^2} \right) - \mathrm{Eu} \frac{\partial p_2}{\partial r} = u_1 \frac{\partial u_1}{\partial r} + \frac{v_1}{r} \frac{\partial u_1}{\partial r} - \frac{v_1^2}{r} \, ;$$

$$\frac{1}{\mathrm{Re}} \left(\nabla^2 v_2 + \frac{2}{r^2} \frac{\partial u_2}{\partial \varphi} - \frac{v_2}{r^2} \right) - \mathrm{Eu} \frac{1}{r} \frac{\partial p_2}{\partial \varphi} = u_1 \frac{\partial v_1}{\partial r} + \frac{v_1}{r} \frac{\partial v_1}{\partial r} - \frac{u_1 v_1}{r} \, ;$$

$$\frac{1}{\mathrm{Re}} \nabla^2 w_2 - 2 (v_1 \cos \varphi + u_1 \sin \varphi) = 0 \, ; \quad \frac{\partial}{\partial r} (r u_2) + \frac{\partial v_2}{\partial \varphi} = 0 \, .$$

From the first two equations of this system it follows that, to a second approximation, the change in the transverse velocity component of the flow is due to the action of the convective acceleration on the flow. The Coriolis



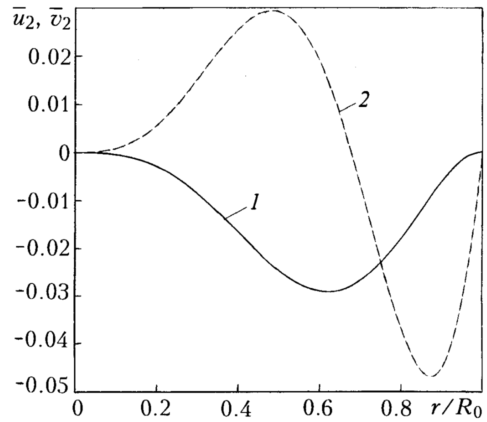


Fig. 3. Pattern of flow in the cross section of a pipe for different values of the stream function $\overline{\psi}_2 = \frac{\psi_2}{\mathrm{Re}}\left(\frac{24\nu}{\omega R_0^2}\right)^2$.

Fig. 4. Profiles of radial (1) and transversal (2) velocities of transverse flow in the second approximation.

force acts directly only on the longitudinal velocity component, decreasing it. The solution of the system of equations of the second approximation has the form

$$\varepsilon^2 u_2 = \frac{\mathrm{Re}}{5\cdot 24^3}\left(\frac{\omega R_0^2}{\nu}\right)^2 r^3\,(r^6 - 4r^4 + 5r^2 - 2)\cos 2\varphi\,;$$

$$\varepsilon^2 v_2 = \frac{\mathrm{Re}}{5\cdot 24^3}\left(\frac{\omega R_0^2}{\nu}\right)^2 r^3\,(-\,5r^6 + 16r^4 - 15r^2 + 4)\sin 2\varphi\,;$$

$$\varepsilon^2 w_2 = \frac{1}{144}\left(\frac{\omega R_0^2}{\nu}\right)^2 \left[r^6 - 3r^4 + 3r^2 - 1 + \frac{r^2}{4}\,(3r^4 - 8r^2 + 5)\cos 2\varphi\right];$$

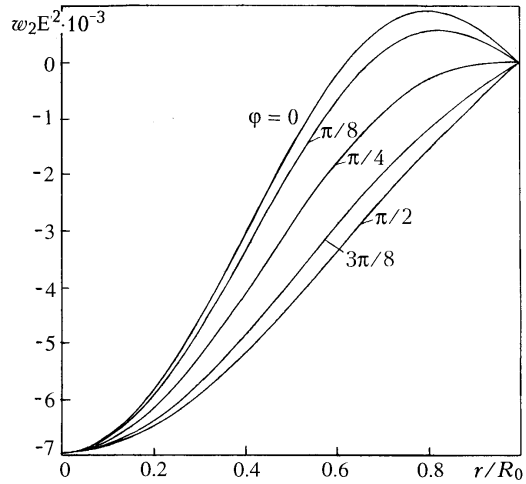
$$\varepsilon^2 p_2 = \frac{r^2}{24^2\,\mathrm{Eu}}\left(\frac{\omega R_0^2}{\nu}\right)^2 \left[\frac{r^2}{3}\,(3r^4 - 8r^2 + 6) + \left(\frac{16}{10}r^6 - 5r^4 + \frac{16}{3}r^2 - 2.1\right)\cos 2\varphi\right].$$

Transverse motion of the medium is described by the stream function

$$\varepsilon^2 \psi_2 = \frac{\mathrm{Re}}{240}\left(\frac{\omega R_0^2}{24\nu}\right)^2 r^4\,(r^6 - 4r^4 + 5r^2 - 2)\sin 2\varphi$$

and represents a vortex system consisting of four vortex formations rotating in the opposite directions and positioned in each quarter of the coordinate space. The vortices positioned in the first and third quarters rotate clockwise, while the vortices positioned in the second and fourth quarters rotate in the opposite direction. The intensity of rotation is proportional to the Re number and the square of the dimensionless quantity $\omega R_0^2/\nu$. Figure 3 shows the streamlines of one such vortex positioned in the first quarter of the coordinate space.

The profiles of the corresponding radial $\overline{u}_2 = \frac{\varepsilon^2 u_2 (24)^3}{\mathrm{Re}\cos 2\varphi}\left(\frac{\nu}{\omega R_0^2}\right)^2$ and transversal $\overline{v}_2 = \frac{\varepsilon^2 v_2 (24)^3}{\mathrm{Re}\sin 2\varphi}\left(\frac{\nu}{\omega R_0^2}\right)^2$ velocities are presented in Fig. 4.



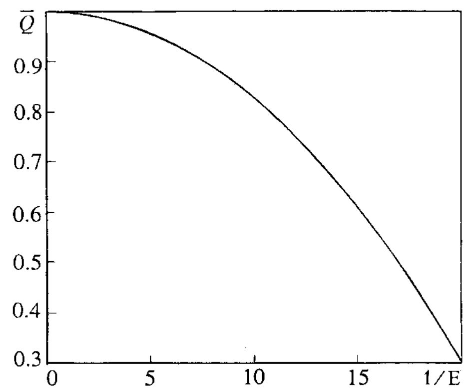


Fig. 5. Profiles of the disturbance of the longitudinal velocity component for different values of the azimuth angle φ.

Fig. 6. Dependence of the volumetric flow rate of the fluid $\overline{Q} = Q/(\pi R_0^2 V_0)$ on the Ekman parameter.

The disturbance of the longitudinal velocity component is described by two functions, one of which is axially symmetric and the second of which is periodically changing with the angular coordinate. The symmetric part of the solution represents fluid motion having zero friction stress on the pipe wall and directed oppositely to the main flow. The velocity profile of this symmetric component of the secondary flow is shown in Fig. 5 (the curve with an azimuth angle of $\varphi = \pi/4$). The asymmetric part represents longitudinal shear flows changing periodically with the angle φ with the zero total flow rate of the fluid through the cross section of the pipe. The disturbance of the velocity in the longitudinal direction is proportional to the square of the dimensionless parameter $\omega R_0^2/\nu$. Figure 5 also shows the graphs of the profile of the longitudinal component of the velocity of disturbed motion in the second approximation for different values of φ. As a result of the interaction of this flow with the main flow, the total profile of the longitudinal velocity also changes periodically with the angle φ. In this case, the velocity of the fluid at the center of the pipe decreases and the profile itself becomes less elongated. As a result, with increase in the parameter $\omega R_0^2/\nu$ the volumetric flow rate of the fluid through the cross section of the pipe decreases by the square law

$$Q = \pi R_0^2 V_0 \left[1 - \frac{1}{576} \left(\frac{\omega R_0^2}{\nu} \right)^2 \right]. \tag{4}$$

The graph of change in the dimensionless flow rate $\overline{Q} = Q/(\pi R_0^2 V_0)$ versus the Ekman parameter $E = \nu/(\omega R_0^2)$ is presented in Fig. 6.

The decrease in the volumetric flow rate of the fluid through the cross section of the pipe is due to the increase in its drag coefficient, which increases as a function of the parameter $\omega R_0^2/\nu$ according to the expression

$$\xi = \frac{64}{\mathrm{Re}} \sqrt{1 + \frac{1}{36\pi^2} \left(\frac{\omega R_0^2}{\nu} \right)^2} .$$

In conclusion, we evaluate the influence of the rotation of the earth on fluid motion in a pipe, using the above relations. From formulas (3), after simple calculations we obtain that with regard for the terms of the first approximation the maximum value of the transverse velocity component is equal to $V_{max} \approx 0.04 \omega R_0^2/\nu$.

For water flowing in a circular pipe with a radius of $R_0 = 0.5$ m, the Ekman parameter is $E = \nu/(\omega R_0^2) = 0.55$. Consequently, $V_{max} \approx 0.07$.

As is seen, the disturbance of the transverse velocity component is about 7% of the mean velocity of the initial undisturbed fluid flow. The hydrodynamic drag increases by 0.5% and the volumetric flow rate of the fluid through the cross section decreases by 0.6% in accordance with (4). Thus, according to the calculations presented, the rotation of the earth exerts a fairly marked influence on the change in the hydrodynamic characteristics of the fluid flow in pipes, which can occur in actual practice, for example, in the case of transportation of a liquid or gas in main pipelines.

NOTATION

$\mathrm{Eu} = p/\rho V_0^2$, Euler number; $\mathrm{Re} = V_0 R_0/\nu$, Reynolds number; $E = \nu/\omega R_0^2$, Ekman parameter; $V_0/\omega R_0$, Rossby number; R_0, radius of the pipe; V_0, mean volume velocity of fluid flow; ν, coefficient of kinematic viscosity; ρ, density; p, pressure; u, v, and w, dimensionless radial, transversal, and longitudinal velocity components, respectively, in the cylindrical coordinate system, assigned to the mean volume velocity of flow V_0; ω, modulus of the vector of the angular velocity of rotation of the coordinate system directed along the OX axis; r, φ, z, dimensionless cylindrical coordinates of the points, assigned to the radius of the pipe.

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